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Self-similar solutions of certain coupled integrable systems

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Abstract
Similarity reductions of the coupled nonlinear Schrödinger equation and an integrable version of the coupled Maxwell–Bloch system are obtained by applying non-translational symmetries. The reduced system of coupled ordinary differential equations are solved in terms of Painlevé transcendents, leading to new exact self-similar solutions for these integrable equations.

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1. Introduction

Completely integrable systems play an important role in many physical applications including water waves, plasma physics, field theory and nonlinear optics. An important feature of many integrable evolution equations is that a large class of their exact solutions, particularly the solitons, can be derived by applying the method of inverse scattering transform (IST) (see [1, 2] for a review). Another significant characteristic shared by many (perhaps all) integrable partial differential equations (PDEs) is that their dimensional reductions to ordinary differential equations (ODEs) have solutions with *no* movable critical points in appropriate variables. This remarkable property is known as the Painlevé property and the corresponding ODEs are said to be of 'Painlevé type'. The relationship between integrability and Painlevé property motivated Ablowitz, Ramani and Segur [3] to make the conjecture (hereafter referred to as the ARS conjecture) that a nonlinear evolution equation is solvable by IST only if *every* ODE obtained by exact similarity reduction is of Painlevé type, perhaps after a transformation of variables. There is considerable evidence that the ARS conjecture is true despite the absence of a complete proof at the present time. Nonetheless, this conjecture provides a preliminary test that is useful to determine whether a PDE is integrable. Conversely, failure of the conjecture in a certain case would strongly suggest that the given PDE is not solvable via IST. Besides playing an important role in the identification of integrable PDEs, the dimensional reduction of

PDEs also provides an effective method of obtaining special classes of exact solutions. Indeed, a large number of self-similar solutions have been found for equations that are solvable by IST by applying the dimensional reduction technique. Typically, these solutions are expressible in terms of elliptic functions, Painlevé transcendents or their degenerations (see, e.g. [2] and references therein). The similarity reduction method can even be applied to obtain special solutions of non-integrable equations.

In this paper, we study the dimensional reductions of certain completely integrable equations which have important applications in nonlinear optics. The equations considered here are the two-component generalizations of the Maxwell–Bloch (MB) and the nonlinear Schrödinger equations. In the text, they are referred to as the coupled Maxwell–Bloch (CMB) equations describing the propagation of ultra-short laser pulses in a resonant medium of three-level atoms, and the coupled nonlinear Schrödinger (CNLS) equations modelling optical pulses in a birefringent optical fibre supporting two linearly polarized propagation modes. The travelling wave and N-soliton solutions obtained by IST and Bäcklund transformations have already been discussed in the literature for both the CMB [4, 5] and the CNLS [6–8] equations. In this work, we construct new self-similar solutions of the CMB and CNLS equations obtained by imposing non-translational symmetries on the original PDEs. It is possible that these new solutions may be of interest in future applications such as optical systems with memory [9]. Moreover, as the original MB and NLS equations, the obtained self-similar solutions of the reduced CMB and CNLS equations are also found to be Painlevé type and are in fact given in terms of the Painlevé transcendents. But in spite of the similarity in the analysis of the Lie-point symmetries, between the coupled systems and the scalar MB and NLS equations, the solution process is more involved in the coupled case because of the increased number of degrees of freedom.

The paper has two main sections. In section 2, we consider the MB and the CMB equations. After introducing the requisite mathematical framework for these equations, we discuss their scaling reductions. The MB equations are treated first as a special case of the CMB equations. We show that the reduced system of ODEs can be interpreted as a Hamiltonian dynamical system with a time-dependent Hamiltonian. Then these reduced ODEs for both the MB and CMB equations are explicitly solved in terms of the third Painlevé transcendent PIII. The similarity reductions of the NLS and CNLS equations are discussed next in section 3. These equations admit two types of non-translational symmetries, namely Galilean boost and scaling. For both the NLS and CNLS equations, the invariant solutions are found in terms of the second Painlevé transcendent PII in the case of Galilean boost, whereas for the scaling symmetry the corresponding invariant solutions are described by the fourth Painlevé transcendent PIV.

2. The coupled Maxwell–Bloch equations

The Maxwell–Bloch (MB) equations describe the propagation of ultra-short optical pulses in a coherent medium of two-level atoms and arise in the study of self-induced transparency [10]. In the lossless case, the MB equations are integrable via the IST method even when the inhomogeneous broadening of the medium is taken into account [11]. In recent years, there also have been considerable theoretical and experimental interests in the propagation of a pair of matched pulses through an absorbing medium of three-level atoms [12–14]. These studies have applications in quantum coherence and interference phenomena (generated by two photon transitions) such as electromagnetically induced transparency [15], lasing with inversion [16] and production of high refractive index materials [17].

2.1. Mathematical background

In the slowly varying amplitude and lossless approximations, the spatio-temporal dynamics of a pulse pair in a three-level resonant medium is given by [12, 13, 5] the Schrödinger equations

$$\frac{\partial a_1}{\partial t} = \mathrm{i}\Omega_1 a_3 \qquad \frac{\partial a_2}{\partial t} = \mathrm{i}\Omega_2 a_3 \qquad \frac{\partial a_3}{\partial t} = \mathrm{i}(\overline{\Omega}_1 a_1 + \overline{\Omega}_2 a_2) \tag{2.1a}$$

and the Maxwell equations

$$\frac{\partial \Omega_j}{\partial z} + \frac{1}{v}\frac{\partial \Omega_j}{\partial t} = -2\mathrm{i}\mu_j a_j \overline{a}_3 \qquad j = 1, 2. \tag{2.1b}$$

In the above equations, a_k is the probability amplitude of the atomic level $|k\rangle$, $k = 1, 2, 3$, and Ω_j, $j = 1, 2$ denote the (normalized) complex electromagnetic field amplitudes also called the Rabi frequencies corresponding to $|j\rangle - |3\rangle$ transitions. Here and throughout the rest of this paper, overbar indicates complex conjugate. Furthermore, only one-dimensional pulse propagation is considered in equation (2.1*b*) where both pulses are travelling along the z-direction with velocity v, and where μ_j, $j = 1, 2$ are the propagation coefficients assumed to be equal (i.e., $\mu_1 = \mu_2 = \mu \neq 0$). For three-level systems, there are only two known integrable cases which correspond to the Λ and V configurations of the atomic energy levels. In above, $|3\rangle$ is the ground state for the V system whereas for the Λ system $|3\rangle$ corresponds to the highest excited state.

It is useful to introduce pulse-local coordinates: $\tau = z$, $x = t - z/v$, and to represent the Schrödinger–Maxwell equations (2.1*a*), (2.1*b*) in the form

$$U_\tau = [\rho, J] \qquad \rho_x = [U, \rho] \tag{2.2a}$$

where subscripts denote partial derivatives, and the 3×3 matrices U, ρ and J are defined as

$$U \equiv \begin{pmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -\overline{u}_1 & -\overline{u}_2 & 0 \end{pmatrix} \qquad u_j = \mathrm{i}\Omega_j \quad j = 1, 2$$
$$J \equiv \mathrm{diag}(-\mu, -\mu, \mu) \qquad [\rho]_{ij} \equiv a_i \overline{a}_j. \tag{2.2b}$$

ρ is the probability density matrix and has vanishing determinant. The second equation in (2.2*a*) for the matrix elements of ρ is called the Bloch equation. We refer to equations (2.2*a*) and (2.2*b*) as the coupled Maxwell–Bloch (CMB) equations. When $a_1 = u_1 = 0$ (or $a_2 = u_2 = 0$) in (2.2*b*), the CMB equations reduce to the MB equations in a medium of two-level atoms.

The system of equations (2.2*a*) can be expressed as the integrability condition of an associated linear system (Lax pair)

$$\Psi_x = (U + \lambda J)\Psi \qquad \Psi_\tau = \frac{\rho}{\lambda}\Psi$$

where λ is the spectral parameter. The Lax-pair and the zero-curvature representation were exploited to obtain special classes of solutions of the CMB equation via IST and Bäcklund transformation techniques (see e.g. [4, 5] and references therein). A field-theoretic description of the CMB equation in terms of symmetric spaces $SU(3)/U(2)$ was given in [18] where the hidden symmetries and conservation laws were also discussed. We remark that the MB and the CMB equations can also be obtained as dimensional reductions of the self-dual Yang–Mills equation with gauge groups $SU(2)$ and $SU(3)$, respectively. (Reductions of the Lax pair for the self-dual Yang–Mills equations giving rise to the above linear system for Ψ were discussed, for example, in [19].) In this paper, our aim is to study the similarity reductions of the CMB equations and to derive its self-similar solutions that are distinct from the travelling wave or breather-type solutions obtained earlier [5]. Furthermore, we wish to investigate whether these self-similar solutions possess the Painlevé property which is a characteristic feature of integrable equations. We first consider the similarity reduction of the MB system which is a special case of the CMB equation, and treat the more general case next.

2.2. Reduction of the Maxwell–Bloch system

If we set $a_1 = \Omega_1 = 0$ (or $a_2 = \Omega_2 = 0$) in equations (2.1*a*), (2.1*b*), then this leads to the Schrödinger–Maxwell equations for a two-level medium. As mentioned above, the corresponding CMB equations then reduce to the MB equations. The resulting equations (2.2*a*) and (2.2*b*) (with $a_2 = u_2 = 0$) can be expressed in the form

$$\begin{aligned} &\hat{U}_\tau = [\hat{\rho}, \hat{J}] \qquad \hat{\rho}_x = [\hat{U}, \hat{\rho}] \\ &\hat{U} \equiv \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \qquad \hat{J} \equiv \mathrm{diag}(-\mu, \mu) \qquad [\hat{\rho}]_{ij} \equiv a_i \overline{a}_j \qquad i, j = 1, 3 \end{aligned} \tag{2.3}$$

and describe the propagation of a single pulse in a two-level resonant medium in the absence of inhomogeneous broadening (sharp line limit). Here we point out that equations (2.3) differ slightly from the usual form of MB equations which appear in the literature (see e.g. [11]) where the trace-free part of $\hat{\rho}$ namely, $\hat{\rho}_0 \equiv \hat{\rho} - \mathrm{Tr}(\hat{\rho})/2$ is used instead of $\hat{\rho}$ itself. However since the trace satisfies $\mathrm{Tr}(\hat{\rho})_x = 0$ (which leads to the conservation of probability density with suitable initial conditions), it follows that (2.3) is equivalent to the MB equations with $\hat{\rho}$ replaced by $\hat{\rho}_0$. It is also worth noting that if the field $u(x, 0)$ is real (or has constant phase), then $u(x, \tau)$ stays real (or its phase remains constant). In this case, equation (2.3) is equivalent to the sine-Gordon equations, which has similarity reductions to the third Painlevé equation (see e.g. [2]).

A class of self-similar solutions of the MB equations arise from the investigation of Lie-point symmetries which leave equation (2.3) invariant. Besides the translational symmetry which leads to travelling wave solutions, the MB equations admit a one-parameter subgroup of scaling symmetry: $x \to \varepsilon^{-1}x$, $\tau \to \varepsilon\tau$, $\hat{U} \to \varepsilon\hat{U}$ and $\hat{\rho} \to \hat{\rho}$. The solutions invariant under the scaling symmetry have the form

$$\hat{U}(x, \tau) = \hat{Q}(\xi)/x \qquad \hat{\rho}(x, \tau) = \hat{\rho}(\xi) \qquad \xi = \sqrt{x\tau}.$$

Substituting the above form of $\hat{U}$ and $\hat{\rho}$ into equation (2.3) yields the following set of coupled ODEs for the off-diagonal matrix elements $q, \bar{q}$ of $\hat{Q}$ and $\hat{\rho}_{ij}$

$$\begin{aligned} &q' = 4\xi\mu\hat{\rho}_{13} \qquad \xi\hat{\rho}'_{13} = 2q(\hat{\rho}_{33} - \hat{\rho}_{11}) \\ &\xi\hat{\rho}'_{11} = -\xi\hat{\rho}'_{33} = 2(q\hat{\rho}_{31} + \bar{q}\hat{\rho}_{13}) \qquad \hat{\rho}_{31} \equiv \overline{\hat{\rho}_{13}} \end{aligned} \tag{2.4}$$

where prime indicates derivative with respect to the argument. The system of ODEs (2.4) admit the first integrals

$$\hat{\rho}_{11} + \hat{\rho}_{33} = \mathrm{Tr}(\hat{\rho}) = a \qquad q\hat{\rho}_{31} - \bar{q}\hat{\rho}_{13} = \mathrm{Tr}(\hat{Q}\hat{\rho}) = b \tag{2.5}$$

where a and b are constants and $a \neq 0$. Making use of the first integrals and the fact that $\det\hat{\rho} = \hat{\rho}_{11}\hat{\rho}_{33} - \hat{\rho}_{13}\hat{\rho}_{31} = 0$, it is possible to reduce (2.4) to a system of two first-order ODEs

$$\begin{aligned} &y' - 4\mu + \frac{y}{\xi} = 2y^2(\hat{\rho}_{11} - \hat{\rho}_{33}) \qquad \text{where} \quad y = \frac{q}{\hat{\rho}_{13}\xi} \\ &(\hat{\rho}_{11} - \hat{\rho}_{33})' = 2y[a^2 - (\hat{\rho}_{11} - \hat{\rho}_{33})^2] - \frac{4b}{\xi}. \end{aligned}$$

If we eliminate the quantity $\hat{\rho}_{11} - \hat{\rho}_{33}$ from the equations immediately above and make the change of variables

$$W = \lambda y \qquad Z = k\xi \qquad \text{such that} \quad \lambda k = a \quad \text{and} \quad k\lambda^{-1} = 2\mu$$

in the resulting second-order ODE for y, then we obtain the third Painlevé equation PIII for $W(Z)$ [20] (p 335, equation XIII)

$$\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} = \frac{1}{W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 - \frac{1}{Z}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right) + \frac{\alpha W^2 + \beta}{Z} + \gamma W^3 + \frac{\delta}{W} \tag{2.6}$$

with the choice of parameters $\alpha = -8b/a$ and $\beta = \gamma = -\delta = 4$. Note that in the present context $W(z)$ represents the solution of a one-parameter family of PIII transcendents since three of the four parameters are fixed. (In general, only two of the four parameters in PIII are free, the remaining two parameters can be fixed by suitable re-scalings of the dependent and independent variables.) Finally, working backward from (2.6) it is straightforward to show that the scale-invariant solutions of the MB equations given by equation (2.4) can be expressed in terms of $W(Z)$ and $W'(Z)$. We omit the details.

It is also possible to derive the PIII reduction of the MB equations by a slightly different route which actually leads to the fifth Painlevé equation PV with special parameter values. This special PV equation can be reduced to PIII by a known transformation [21]. We outline this approach here since it is also relevant to the similarity reduction of the CMB equations to be discussed next. If we use the variable $\hat{g} \equiv \hat{\rho}_{11}$ instead of y, then by differentiating the equation for $\hat{\rho}_{11}$ in (2.4) we get

$$\hat{g}'' = \left(\frac{\hat{g}'^2}{2} - \frac{2b^2}{\xi^2}\right)\left(\frac{1}{\hat{g}} - \frac{1}{a - \hat{g}}\right) - \frac{\hat{g}'}{\xi} + 16\mu\hat{g}(a - \hat{g}).$$

To derive the above equation for $\hat{g}$, we used the remaining equations from (2.4), the first integrals from (2.5) and the relation $\hat{\rho}_{11}\hat{\rho}_{33} = \hat{\rho}_{13}\hat{\rho}_{31}$ (i.e. $\det\hat{\rho} = 0$). Next, by introducing the variable $\hat{y} = (\hat{g} - a)/\hat{g}$ we can further transform the above ODE to

$$\hat{y}'' = \left(\frac{1}{2\hat{y}} + \frac{1}{\hat{y} - 1}\right)\hat{y}'^2 - \frac{\hat{y}'}{\xi} + \frac{4(\hat{y} - 1)^2}{\xi^2}\left(\alpha\hat{y} + \frac{\beta}{y}\right) + 2\gamma\hat{y} \tag{2.7}$$

where $\alpha = b^2/2a^2 = -\beta$ and $\gamma = -8\mu a$. It is equation (2.7) that can either be transformed to a special PV equation with two free parameters or to a special PIII equation with one free parameter, thereby inducing a transformation between the special cases of PIII and PV themselves. We defer the details of these transformations to the next subsection where they will be discussed in a more general context.

2.3. Reduction of the coupled Maxwell–Bloch equations

As the MB equations, the CMB equations also admit ODE reductions under the scaling symmetry. The scaling-invariant solutions are obtained from (2.2*a*) by expressing the 3×3 matrices in (2.2*b*) in the form

$$U(x, \tau) = Q(\xi)/x \qquad \rho(x, \tau) = \rho(\xi) \qquad \xi = \sqrt{x\tau}$$

where the matrices $Q(\xi)$ and $\rho(\xi)$ satisfy

$$\xi\rho' = 2[Q, \rho] \qquad Q' = 2\xi[\rho, J]. \tag{2.8}$$

Alternatively, one can directly consider the scaling reductions of the Schrödinger–Maxwell equations (2.1*a*), (2.1*b*) by expressing the electromagnetic fields and the probability amplitudes as

$$\mathrm{i}\Omega_j(z, t) = u_j(x, \tau) = q_j(\xi)/x \qquad j = 1, 2$$

$$a_k(z, t) = a_k(x, \tau) = a_k(\xi) \qquad k = 1, 2, 3.$$

The reduced system of ODEs are simpler to express in terms of the variables a_k rather than the Bloch matrix elements $\rho_{ij} = a_i \overline{a}_j$ (which are quadratic in the a_k). The ODEs for the variables $q_j(\xi)$, $j = 1, 2$ and $a_k(\xi)$, $k = 1, 2, 3$ are given by

$$q_j' = 4\xi\mu a_j \overline{a}_3 \qquad \xi a_j' = 2q_j a_3 \qquad j = 1, 2 \qquad \xi a_3' = -2(\overline{q}_1 a_1 + \overline{q}_2 a_2). \tag{2.9}$$

Equations (2.9) together with the complex conjugate equations imply the matrix ODE system (2.8). The reduced ODEs admit the following first integrals

$$\begin{aligned} &\sum_{k=1}^{3} |a_k|^2 = \mathrm{Tr}(\rho) = A \qquad \sum_{j=1}^{2} (a_3 q_j \overline{a}_j - \text{c.c.}) = \mathrm{Tr}(Q\rho) = B \\ &q_2 a_1 - q_1 a_2 = C \qquad (|C|^2 = \mathrm{Tr}(\rho Q^2 - AQ^2/2)) \end{aligned} \tag{2.10}$$

where A, B, C are constants and c.c. denotes complex conjugate. Note that unlike the reduced MB equations (2.4), the reduced CMB equations (2.9) admit an additional first integral C ($C \equiv 0$ for the MB equations).

It is interesting to note that equations (2.9) and their complex conjugates can be recast as a Hamiltonian dynamical system on a ten-dimensional (real) phase space with variables q_j, $j = 1, 2$ and $b_k \equiv \xi a_k$, $k = 1, 2, 3$ together with their complex conjugates. When expressed in terms of the new variables b_k, equations (2.9) become

$$\xi q_j' = 4\mu b_j \overline{b}_3 \qquad \xi b_j' = b_j + 2q_j b_3 \qquad j = 1, 2 \qquad \xi b_3' = b_3 - 2(\overline{q}_1 b_1 + \overline{q}_2 b_2). \tag{2.9$'$}$$

Equations (2.9′) represent a Hamiltonian system prescribed by the canonical Poisson bracket relations and a time (ξ)-dependent Hamiltonian function H as follows:

$$q_j' = \{q_j, H\} \qquad j = 1, 2 \qquad b_k' = \{b_k, H\} \qquad k = 1, 2, 3$$

$$H = \frac{1}{\xi}\left(\sum_{k=1}^{3} |b_k|^2 + 2\sum_{j=1}^{2} (b_3 q_j \overline{b}_j - \text{c.c.})\right) = \xi\, \mathrm{Tr}(\rho + 2Q\rho).$$

The associated Poisson structure is defined by the fundamental Poisson brackets

$$\{q_i, \overline{q}_j\} = -2\mu\delta_{ij} \qquad i, j = 1, 2 \qquad \{a_k, \overline{a}_l\} = \delta_{kl} \qquad k, l = 1, 2, 3$$

where δ_{ij} is the usual Kronecker delta, and all other fundamental Poisson brackets vanish. Furthermore, the first integrals in equation (2.10) are in involution with respect to the Poisson structure defined above.

The next task is to solve the system (2.9) and to determine whether the general solution has the Painlevé property. The similarity reductions of the CMB equations are expected to be of Painlevé type because of the ARS conjecture. However to our knowledge, the Painlevé property of these ODEs has not yet been studied. In what follows, we show that there exists a suitable combination of variables (not necessarily unique) that satisfies the third Painlevé equation PIII in general position. Secondly, the CMB variables in (2.9) can be expressed via the PIII transcendent either by quadratures or through the solution of a Riccati equation.

It is useful for subsequent calculations to introduce the variables

$$g \equiv |a_1|^2 + |a_2|^2 = A - |a_3|^2 \qquad \text{and} \qquad f \equiv a_3(\overline{a}_1 q_1 + \overline{a}_2 q_2).$$

Note that the first integral B in equation (2.10) can be written as $B = f - \bar{f}$. Starting from equation (2.9) and making use of the first integrals (2.10), a straightforward computation yields

$$\begin{aligned} &\xi g' = 4f - 2B \qquad \xi f' = 4\mu\xi^2 g(A - g) + 2h(A - 2g) + 2|C|^2 \\ &\text{where} \qquad h \equiv |q_1|^2 + |q_2|^2 = \frac{f(f - B)}{g(A - g)} + \frac{|C|^2}{g}. \end{aligned} \tag{2.12}$$

Eliminating $f(\xi)$ from the two first-order equations (2.12), we obtain

$$g'' = \left(\frac{g'^2}{2} - \frac{2B^2}{\xi^2}\right)\left(\frac{1}{g} - \frac{1}{A-g}\right) - \frac{g'}{\xi} + \frac{8|C|^2}{\xi^2}\left(\frac{A-g}{g}\right) + 16\mu g(A-g)$$

which is rational in $g(\xi)$ and $g'(\xi)$, and is of Painlevé type. Indeed, by the rational substitution $y = (g - A)/g$, the above ODE for $g(\xi)$ can be transformed into the following intermediate form

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{\xi} + \frac{4(y-1)^2}{\xi^2}\left(\alpha y + \frac{\beta}{y}\right) + 2\gamma y \tag{2.13}$$

where $\alpha = (B^2 - 4A|C|^2)/2A^2$, $\beta = -B^2/2A^2$ and $\gamma = -8\mu A$. Here we note that when $C = 0$, equation (2.13) reduces to equation (2.7) of the previous subsection. This corresponds to the special cases of the CMB equations when either (a) $q_1/q_2 = a_1/a_2$, $q_j \neq 0$, $a_j \neq 0$, $j = 1, 2$ or (b) $q_j = a_j = 0$, $j = 1$ or 2. The latter case (b) is simply the degeneration of the CMB equations to the MB equations.

Changing the variables to $Z = \xi^2/2$, $W(Z) = y(\xi)$ in equation (2.13), leads to a special case of the fifth Painlevé equation PV [20] (p 341, equation XXXIX, $\delta = 0$)

$$\frac{\mathrm{d}^2W}{\mathrm{d}Z^2} = \left(\frac{1}{2W} + \frac{1}{W-1}\right)\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 - \frac{1}{Z}\frac{\mathrm{d}W}{\mathrm{d}Z} + \frac{(W-1)^2}{Z^2}\left(\alpha W + \frac{\beta}{W}\right) + \frac{\gamma W}{Z} \tag{2.14}$$

with the same parameters α, β, γ as defined above. There also exists a correspondence between $y(\xi)$ in equation (2.13) and solutions of PIII in general position (i.e. with two free parameters). This transformation [21] between solutions of PIII and the special case of PV with $\delta = 0$ is given as follows. Define the function $\widehat{W}(\xi)$ via the solution $y(\xi)$ of equation (2.13) by

$$\widehat{W} = -\frac{y'}{2y} + \frac{c_1 y}{\xi} + \frac{c_2}{y\xi} - \frac{c_1 + c_2}{\xi}$$

where $c_1^2 = 2\alpha$ and $c_2^2 = -2\beta$ are given in terms of the parameters α, β of equation (2.13). Then $\widehat{W}(\xi)$ is a solution of PIII and satisfies equation (2.6) with parameters $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ where $\hat{\alpha} = 2(c_1 + c_2)$, $\hat{\beta} = 2(c_1 - c_2 - 1)/\gamma$, $\hat{\gamma} = 1$ and $\hat{\delta} = -\gamma^2$.

Thus we have shown how to express $y(\xi)$ or equivalently, $g(\xi) = |a_1|^2 + |a_2|^2$ in terms of PV (or PIII) transcendent. In fact, it turns out that the solution of the full system of ODEs (2.9) can be obtained in terms of this Painlevé transcendent and its derivative, as outlined below. Given $g(\xi)$ we solve for $f(\xi)$ from the first equation in (2.12). Next, from (2.9) we obtain

$$Y_j' = 2(g - A)Y_j^2 - \frac{2fY_j}{g - A} + 4\xi\mu$$

$$\text{where} \qquad Y_j = \frac{q_j}{a_j\overline{a}_3} \quad j = 1, 2 \qquad \text{and} \qquad \frac{\overline{a}_3'}{\overline{a}_3} = \frac{2f}{g - A}.$$

The last equation above can be solved for $\overline{a}_3$. Then from two independent solutions Y_1 and Y_2 of the Riccati equation, the CMB variables q_j and a_j, $j = 1, 2$ can be obtained in quadratures as follows:

$$\frac{q_j'}{q_j} = \frac{4\xi\mu}{Y_j} \qquad a_j = \frac{q_j}{\overline{a}_3 Y_j} \qquad j = 1, 2.$$

The remaining variables $\overline{q}_j$, $\overline{a}_j$ and a_3 can be obtained in a similar fashion, starting with $g(\xi)$ and $\bar{f}(\xi) = f(\xi) - B$. Thus all the CMB variables in (2.9) can be obtained in terms of g and f either by quadratures or by solving Riccati equations.

We conclude this section by remarking that the similarity reductions of the Maxwell–Bloch equations for two- and three-level atomic media can be extended to the multi-level case of n-CMB equation describing n electromagnetic fields propagating in a resonant medium with $n+1$ energy states for $n > 2$. This can be achieved in a relatively straightforward manner by appropriate generalizations of the equations (2.8)–(2.10). Besides $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}(Q\rho)$, there are $n(n-1)/2$ additional first integrals given by $C_{ij} = q_i a_j - q_j a_i$, $i \neq j$, $i, j = 1, 2, \ldots, n$. By introducing the variable $\tilde{g}(\xi) \equiv |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2$ and proceeding in the same way as we did for the CMB reduction, the scaling-invariant solutions of the n-CMB equations are also obtained via the general PIII transcendent. We do not go into further details of the n-CMB case in this paper.

3. The coupled nonlinear Schrödinger equations

In this section we study the similarity reductions of a system of two coupled NLS (CNLS) equations in 1 + 1 dimensions. This system is known to be completely integrable in the sense that it admits soliton solutions and the initial value problem can be solved via the IST method [6]. The CNLS equations also possess an infinite set of conserved quantities [7], and pass the Painlevé PDE test [8]. Thus, it is reasonable to expect that in accord with the ARS conjecture, the ODEs obtained by the dimensional reductions of the CNLS equations are of Painlevé type. Indeed in [22, 23], a set of Painlevé-type ODEs regarded as a coupled system of Painlevé II equations was derived by reducing the CNLS equations by the Galilean boost symmetry. In this section we derive a more general set of coupled ODEs and show that their solutions are given directly via the second Painlevé transcendent. We also present another dimensional reduction of the CNLS equations under the scaling symmetry and obtain the invariant solutions in terms of the fourth Painlevé transcendent.

3.1. Reductions of NLS equations

Here we review briefly the known similarity reductions of the NLS equation

$$\mathrm{i}u_t = \frac{u_{xx}}{2} + s|u|^2 u \qquad (3.1)$$

for the complex function $u(x, t)$. We consider both focusing ($s = 1$) and defocussing ($s = -1$) cases. Analysis of the classical Lie-point symmetries of equation (3.1) leads to three types of similarity reductions (see e.g. [24]) namely, translation, scaling and Galilean boost. The translational symmetry leads to the well-known travelling wave solutions. In this paper, we discuss the non-translational symmetries only. There are two cases that we consider.

Case 1. Galilean boost [24]: in this case the form of the field $u(x, t)$ is given by

$$u(x,t) = r(z)\exp(\mathrm{i}\phi(z) - \mathrm{i}\alpha t(x - \alpha t^2/3)) \qquad z = x - \alpha t^2/2 \qquad (3.2)$$

where $\alpha > 0$ is a constant parameter. Substituting (3.2) into equation (3.1) and separating real and imaginary parts, we find that $r(z)$ and $\phi(z)$ satisfy the system of equations

$$r\phi'' + 2r'\phi' = 0 \qquad (3.3a)$$

$$r'' - r\phi'^2 + 2sr^3 - 2\alpha zr = 0. \qquad (3.3b)$$

Equation (3.3a) may be integrated once to obtain $\phi' = C/r^2$, where C is the constant of integration. Substituting the result into (3.3b) leads to a non-autonomous ODE for $r(z)$

$$r'' = C^2 r^{-3} - 2sr^3 + 2\alpha zr. \qquad (3.4)$$

If $C = 0$, equation (3.4) can simply be rescaled to

$$\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} = 2W^3 + ZW \qquad \text{where} \quad W = (-s)^{1/2}(2\alpha)^{-1/3} r \quad Z = (2\alpha)^{1/3} z \qquad (3.5)$$

which is a special case of the second Painlevé equation PII [20] (p 334, equation IX with $\gamma = 0$). This special case of PII has also been obtained as the Galilean boost reduction of the NLS equation with external potential [25].

When $C \neq 0$, equation (3.4) is still solvable in terms of a Painlevé-type ODE. By an appropriate change of variables in equation (3.4), it is possible to derive the following equation

$$\frac{d^2W}{dZ^2} = \frac{1}{2W}\left(\frac{dW}{dZ}\right)^2 + 4aW^2 - ZW - \frac{1}{2W} \tag{3.6}$$

where

$$W = \frac{(4\alpha)^{1/3}}{2iC}r^2 \qquad Z = -(4\alpha)^{1/3}z \qquad \text{and} \qquad a = \frac{isC}{2\alpha}.$$

Equation (3.6) is also a Painlevé-type ODE (see e.g., [20], p 340, equation XXXIV) which can be transformed to PII

$$\frac{d^2V}{dZ^2} = 2V^3 + ZV + \alpha$$

via the transformation $2aW = V' + V^2 + Z/2$, where the parameter $\alpha = -2a - 1/2$.

Case 2. Scaling reduction: the form of $u(x,t)$ is given by

$$u(x,t) = t^{(-1+i\mu)/2}q(z) \qquad z = xt^{-1/2} \tag{3.7}$$

where the constant μ is real. Substituting (3.7) into (3.1) yields the following ODE for $q(z)$:

$$iq'' = zq' + q - i(\mu + 2s|q|^2)q. \tag{3.8}$$

If one defines $q(z) = r(z)\exp(i\phi(z))$ and proceeds in the same way as in the case of the Galilean boost reduction, then it is possible to eventually obtain a third-order ODE in the nonlocal variable $v(z)$, where $v'(z) = r^2(z)$. The equation for $v(z)$ can be integrated once to obtain a second-order, second-degree ODE which possesses the Painlevé property and whose solution can be expressed in terms of the PIV transcendent and its derivatives [26, 27]. Here we do not give the details of this reduction procedure which can be found in the literature (see e.g. [28, 29]). Instead, we describe a different procedure to obtain the PIV equation more directly from (3.8) and in terms of a *local* variable defined in terms of $q(z)$ and its derivative. From equation (3.8) and its complex conjugate, we construct a first-order system of ODEs

$$b' = 2aq \qquad \bar{b}' = -2a\bar{q} \qquad a' = s(q\bar{b} - \bar{q}b) \tag{3.9a}$$

where $b = -iq' + zq$, $\bar{b}$ is the complex conjugate of b and $2a = i(\mu + 2s|q|^2)$. Note that the above system admits a first integral given by

$$a^2 - s|b|^2 = -m^2 \tag{3.9b}$$

where m is a constant. Next we introduce a new variable $y \equiv b/q = -iq'/q + z$, which is local in $q(z)$ and $q'(z)$ unlike the previously discussed similarity reduction. Then it is a straightforward calculation by using equations (3.9*a*), (3.9*b*) to obtain a second-order ODE for $y(z)$, namely,

$$y'' = \frac{y'^2}{2y} - \frac{3y^3}{2} + 2zy^2 + y(\mu + i - z^2/2) + \frac{2m^2}{y}. \tag{3.10}$$

After the following rescaling of the dependent and independent variables in equation (3.10)

$$y = \beta W \qquad z = kZ \qquad \text{such that} \qquad 2\beta = -k \quad k^2 = 2i$$

we obtain equation PIV [20] (p 339, equation XXXI)

$$\frac{d^2W}{dZ^2} = \frac{1}{2W}\left(\frac{dW}{dZ}\right)^2 + \frac{3W^3}{2} + 4ZW^2 + 2(Z^2 - c)W + \frac{8m^2}{W} \tag{3.11}$$

with $c = -1 + i\mu$.

3.2. Reductions of the CNLS equation

Next we consider the similarity reductions of the CNLS equation

$$\mathrm{i}u_{jt} = \frac{u_{jxx}}{2} + s(|u_1|^2 + |u_2|^2)u_j \qquad j = 1, 2 \tag{3.12}$$

for the pair of complex function $u_j(x, t)$, where $s = 1$ and $s = -1$ are the focusing and defocussing cases, respectively. As in the previous subsection, we consider the two cases of Galilean boost and scaling symmetries.

Case 1. Galilean boost: the CNLS fields $u_j(x, t)$ and the similarity variable are expressed as

$$u_j(x, t) = r_j(z)\exp(\mathrm{i}\phi_j(z) - \mathrm{i}\alpha t(x - \alpha t^2/3)) \qquad z = x - \alpha t^2/2 \quad j = 1, 2 \tag{3.13}$$

where $\alpha > 0$ is a constant. Using the form (3.13) in equation (3.12), we obtain a set of coupled ODEs for the amplitude $r_j(z)$ and the phase $\phi_j(z)$ components

$$r_j\phi_j'' + 2r_j'\phi_j' = 0 \tag{3.14a}$$

$$r_j'' - r_j\phi_j'^2 + 2s\left(r_1^2 + r_2^2\right)r_j - 2\alpha z r_j = 0 \qquad j = 1, 2. \tag{3.14b}$$

Once again, equation (3.14*a*) can be integrated to obtain $\phi_j' = C_j\big/r_j^2$ where C_j, $j = 1, 2$ are constants of integration. Then equation (3.14*b*) reduces to a coupled non-autonomous system for the r_j given by

$$r_j'' = C_j^2 r_j^{-3} + 2(\alpha z - sg)r_j \qquad j = 1, 2 \tag{3.15}$$

where we have defined $g(z) \equiv r_1^2(z) + r_2^2(z)$.

Equation (3.15) with $C_j = 0$ can be regarded as a coupled system of PII equations in the variables r_1 and r_2. This system was derived in [22] where the authors performed a local analysis (in the neighbourhood of some initial values) which suggested that (3.15) possesses the Painlevé property. Here we show that (3.15) does indeed have the Painlevé property for *any* choice for the constants C_j since equation (3.15) can be explicitly solved in terms of the PII transcendent. To that end, we re-express equation (3.15) in terms of the variables $y_j = r_j^2$ and differentiate the resulting equation once to arrive at

$$y_j''' = 8y_j'(\alpha z - sg) + 4y_j(\alpha - sg') \qquad j = 1, 2. \tag{3.16}$$

Now the basic idea is to regard equation (3.16) as a *linear*, third-order equation for each component $y_j(z)$, with variable coefficients depending on $g(z)$ and $g'(z)$. Thus the solution of equation (3.15) is completely expressible in terms of the solutions of the linear equation (3.16), provided the function $g(z)$ is known. An ODE for $g(z)$ can be found readily if we add the two equations for y_1 and y_2 in (3.16) and introduce the variable $h(z)$ where $h'(z) = g(z) = y_1 + y_2$. After integrating the resulting equation once, we obtain

$$h''' + 6sh'^2 - 8\alpha z h' + 4\alpha h - \beta = 0 \tag{3.17}$$

where β is the constant of integration. We remark at this point that the reduction procedure outlined thus far can be easily extended to the n-CNLS equation with complex fields u_j, $j = 1, 2, \ldots, n, n > 2$. This is achieved by appropriately generalizing equations (3.12)–(3.16) for n components and by defining the function $g(z)$ as $g(z) \equiv y_1 + y_2 + \cdots + y_n$.

Equation (3.17) is of Painlevé type and belongs to the third order, polynomial class of Painlevé-type equations studied by Chazy [26] and later by Bureau [27]. To make the connection with PII, we note that equation (3.17) admits a first integral which is of second order and second degree, namely

$$h''^2 + 4sh'^3 - 8\alpha z h'^2 + 8\alpha h h' - 2\beta h' = \gamma$$

where γ is an integration constant. Two of the three constants α, β, γ can be absorbed by the following transformation of variables

$$\sigma_{II} = sk(h + \beta k^3) \qquad Z = z/k \qquad k = -(4\alpha)^{-1/3}$$

in the above equation which then takes the form

$$\left(\frac{\mathrm{d}^2\sigma_{II}}{\mathrm{d}Z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_{II}}{\mathrm{d}Z}\right)^3 + 2Z\left(\frac{\mathrm{d}\sigma_{II}}{\mathrm{d}Z}\right)^2 - 2\sigma_{II}\frac{\mathrm{d}\sigma_{II}}{\mathrm{d}Z} - \hat{\gamma} = 0 \tag{3.18}$$

where $16\alpha^2\hat{\gamma} = \gamma$. The transformation of equation (3.18) into the PII equation was discussed in [26, 27]. More recently, it was noted in [30] that $\sigma_{II}(Z)$ satisfying equation (3.18), is the Hamiltonian function for PII, and is also related to the τ-function simply by $(\ln \tau_{II})'(Z) = \sigma_{II}$. The Hamiltonian formulation for PII is given explicitly as

$$\begin{aligned} &\frac{\mathrm{d}q}{\mathrm{d}Z} = \frac{\partial\sigma_{II}}{\partial p} \qquad \frac{\mathrm{d}p}{\mathrm{d}Z} = -\frac{\partial\sigma_{II}}{\partial q} \\ &\sigma_{II}(p, q, Z) = \frac{p^2}{2} + p\left(q^2 + \frac{Z}{2}\right) + 2\hat{\gamma}^{1/2}q. \end{aligned} \tag{3.19}$$

After eliminating $p(Z)$ from the first-order system given by equation (3.19), one obtains PII for the function $q(Z)$ with the PII parameter given by $1/2 - 2\hat{\gamma}^{1/2}$. It is also clear from (3.19) that $p(Z)$ depends linearly on $q'(Z)$, so that $\sigma_{II}(Z)$ is a polynomial in $q(Z)$ and $q'(Z)$ with a quadratic dependence on $q'(Z)$. In summary, starting from a solution of PII one constructs $\sigma_{II}(Z)$ which (after a change of variable) provides a solution $h(z)$ of equation (3.17). It is then possible to express the amplitude components r_j of CNLS, in terms of solutions of the linear equation (3.16) whose coefficients $g(z) = h'(z)$ and $g'(z) = h''(z)$ depend on PII and its derivatives.

Case 2. Scaling reduction: in this case we set

$$u_j(x, t) = t^{(-1+\mathrm{i}\mu)/2}q_j(z) \qquad j = 1, 2 \qquad z = xt^{-1/2}$$

in analogy with the scaling reduction of the NLS equation (cf equation (3.7)). Now each component $q_j(z)$ satisfies

$$\mathrm{i}q_j'' = zq_j' + q_j - \mathrm{i}[\mu + 2s(|q_1|^2 + |q_2|^2)]q_j \qquad j = 1, 2. \tag{3.20}$$

Next we introduce the variables

$$b_j = -\mathrm{i}q_j' + zq_j \qquad j = 1, 2 \qquad \text{and} \qquad 2a = \mathrm{i}(\mu + 2s(|q_1|^2 + |q_2|^2))$$

which satisfy a system of first-order ODEs

$$b_j' = 2aq_j \qquad \overline{b}_j' = -2a\overline{q}_j \qquad a' = s\sum_{j=1}^{2}(q_j\overline{b}_j - \overline{q}_jb_j) \tag{3.21}$$

derived from equation (3.8) and its complex conjugate. The system (3.21) admits a first integral given by

$$a^2 - s\sum_{j=1}^{2}|b_j|^2 = -m^2 \tag{3.22}$$

m being a constant. We again note that the scaling reduction can also be extended to the general case of multi-component n-CNLS equation in an obvious manner by setting $j = 1, 2, \ldots, n, n > 2$ in equations (3.20)–(3.22).

In the following we show how to relate the function $a(z)$ to a solution of PIV. To this end, we differentiate the equation for $a(z)$ in (3.21) twice, use the remaining equations in (3.21) and the first integral in (3.22) to obtain a third-order ODE for $a(z)$. This ODE for $a(z)$ can be integrated once by introducing the variable $f(z)$ where $f'(z) = a(z)$. The final result is a third-order ODE in $f(z)$ given by

$$f''' = 6\mathrm{i}f'^2 - z(zf' - f) + 2\mu f' + Cz + 2\mathrm{i}m^2 \tag{3.23}$$

where C is a constant of integration. The change of variables

$$\sigma_{IV} = -\mathrm{i}k(f - sC) \qquad Z = z/k \qquad k^2 = 2(-s)^{1/2}$$

in (3.23) leads to the equation for the Hamiltonian function σ_{IV} associated with PIV, namely,

$$\frac{\mathrm{d}^3\sigma_{IV}}{\mathrm{d}Z^3} = -6\left(\frac{\mathrm{d}\sigma_{IV}}{\mathrm{d}Z}\right)^2 + 4Z\left(Z\frac{\mathrm{d}\sigma_{IV}}{\mathrm{d}Z} - \sigma_{IV}\right) + 4\mu(-s)^{1/2}\frac{\mathrm{d}\sigma_{IV}}{\mathrm{d}Z} - 8sm^2. \tag{3.24a}$$

Moreover, there is a first integral of second order and second degree similar to the previous case of PII reduction (cf equation (3.18)), given by

$$\left(\frac{\mathrm{d}^2\sigma_{IV}}{\mathrm{d}Z^2}\right)^2 = 4\left(Z\frac{\mathrm{d}\sigma_{IV}}{\mathrm{d}Z} - \sigma_{IV}\right)^2 - 4\left(\frac{\mathrm{d}\sigma_{IV}}{\mathrm{d}Z} - \nu_0\right)\left(\frac{\mathrm{d}\sigma_{IV}}{\mathrm{d}Z} - \nu_1\right)\left(\frac{\mathrm{d}\sigma_{IV}}{\mathrm{d}Z} - \nu_2\right) \tag{3.24b}$$

where one of the constants, e.g., ν_0 can be set to 0 without loss of generality, then the remaining parameters are given by $\nu_1 + \nu_2 = (-s)^{\frac{1}{2}}\mu$, $\nu_1\nu_2 = 4sm^2$. Equation (3.24*b*) appears in [30] as well. The Hamiltonian formulation for PIV is described via the Poisson bracket relations

$$\begin{aligned} &\frac{\mathrm{d}q}{\mathrm{d}Z} = \{q, \sigma_{IV}\} \qquad \frac{\mathrm{d}p}{\mathrm{d}Z} = \{p, \sigma_{IV}\} \\ &\sigma_{IV}(q, p, Z) = \frac{2p^2}{q} - p\left(q + 2Z + \frac{\nu_1}{q}\right) + \frac{\nu_2}{2}q \end{aligned} \tag{3.25a}$$

where the fundamental Poisson bracket relation is given by

$$\{p, q\} = -\{q, p\} = q. \tag{3.25b}$$

After eliminating $p(Z)$ from the first-order system (3.25*a*), one finds that $q(Z)$ satisfies PIV. Note also that the PIV Hamiltonian σ_{IV} is rational in $q(Z)$, but depends quadratically on $q'(Z)$ as in the PII case. Consequently, $f(z)$ in (3.23) and in particular, the quantity $f'(z) = a(z) = \mathrm{i}(\mu + 2s(|q_1|^2 + |q_2|^2)/2$ in the reduced CNLS equation (3.20) are rational functions of the PIV transcendent and its derivatives. Therefore, equation (3.20) can be regarded as a second-order *linear* equation for each q_j, with coefficients depending on PIV transcendent and its derivatives. Solutions of this linear ODE yield the CNLS component fields q_j.

4. Conclusion

In this paper, we have studied systems of ODEs obtained from the dimensional reductions of the CMB equations by scaling symmetry and of the CNLS equations by Galilean boost and scaling symmetries. We have derived new exact solutions of these ODEs and discussed their underlying Hamiltonian structures. The invariant solutions of the CMB equations are obtained in terms of the third Painlevé transcendent whereas, for the CNLS equations, the solutions invariant under Galilean boost and scaling are respectively given in terms of the second and fourth Painlevé transcendents. The obvious extensions of the reduction procedures to the multi-component *n*-CMB and *n*-CNLS equations are also indicated. In particular, for a Maxwell–Bloch system with probability density function $|a_j|^2$ for level $|j\rangle$, $j = 1, 2, \ldots, n$, $n \geqslant 1$, the ODE satisfied by the quantity $|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2$ can be transformed to the general PIII equation with two arbitrary parameters. For the reduced *n*-CNLS equations with complex fields q_j, $j = 1, 2, \ldots, n$, $n \geqslant 1$, the intensity function $I = |q_1|^2 + |q_2|^2 + \cdots + |q_n|^2$ turns out to be related to the Painlevé transcendents. The ODEs for I obtained from Galilean boost and scaling symmetries can be transformed respectively to the equations satisfied by the Hamiltonians σ_{II} and σ_{IV} associated with PII and PIV. We have also included a review of the known similarity reductions of the scalar ($n = 1$) MB and NLS equations to the Painlevé equations. These results provide a framework for comparison with the corresponding coupled systems. It is interesting to note that in all cases (considered in this work), the invariant solutions to the n-CMB and the n-CNLS equations are ultimately given independently of n, by the *same* Painlevé equation as the $n = 1$ case. The only difference is that the Painlevé equations associated with the $n > 1$ cases have more free parameters.

We conclude the paper on a slightly speculative note. It is possible to derive each of the Painlevé equations (PI–PVI) as the compatibility condition of a pair of 2×2 linear systems which arise in the study of isomonodromy deformation problems [30]. These linear systems are also obtained from the similarity reductions of 2×2 Lax pairs associated with equations solvable by IST (see e.g. [2]). Our study indicates that there are infinitely many linear systems ($n = 1, 2, \ldots$) which arise from the $(n+1) \times (n+1)$ Lax pairs associated with the n-CNLS and n-CMB equations and whose integrability conditions lead to the PII, PIII and PIV equations. It is conceivable then that each of the remaining Painlevé equations may also be derived from the isomonodromic deformations of not one, but a (denumerable) infinite number of linear systems. We hope to address this issue in future.

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References

[1] Ablowitz M and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)
[2] Ablowitz M and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering (LMS Lecture Note Series vol 149)* (Cambridge: Cambridge University Press)
[3] Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo Cimento* **23** 333–8
[4] Basharov A M, Maimistov A I and Manykin E A 1990 *Sov. Phys.–JETP* **70** 864–71
[5] Park Q-Han and Shin H J 1998 *Phys. Rev.* A **57** 4643–53
[6] Manakov S V 1974 *Sov. Phys.–JETP* **38** 248–53
[7] Zakharov V E and Schulman E I 1982 *Physica* D **4** 270–4
[8] Sahadevan R, Tamizhmani K M and Lakshmanan M 1986 *J. Phys. A: Math. Gen.* **19** 1783–91
[9] Levi D, Menyuk C R and Winternitz P (ed) 1994 *Self-Similarity in Stimulated Raman Scattering* (Montréal: CRM)
[10] McCall S L and Hahn E L 1969 *Phys. Rev.* **183** 457–85
[11] Ablowitz M J, Kaup D J and Newell A C 1974 *J. Math. Phys.* **15** 1852–8
[12] Harris S E 1994 *Phys. Rev. Lett.* **72** 52–5
[13] Eberly J H, Pons M L and Haq H R 1994 *Phys. Rev. Lett.* **72** 56–9
[14] Vemuri G and Vasavada K V 1996 *Opt. Commun.* **129** 379–86
[15] Boller K J, Imamoglu A and Harris S E 1991 *Phys. Rev. Lett.* **66** 2593–6
[16] Harris S E 1989 *Phys. Rev. Lett.* **62** 1033–6
[17] Scully M O 1991 *Phys. Rev. Lett.* **67** 1855–8
[18] Park Q-Han and Shin H J 1998 *Phys. Rev.* A **57** 4621–42

[19] Chakravarty S and Ablowitz M J 1992 *Painlevé Transcendents* ed D Levi and P Winternitz (New York: Plenum) pp 331–43
[20] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[21] Lukashevich N A 1967 *Diff. Eqns.* **3** 994–9
[22] Baumann G, Glöckle W G and Nonnenmacher T F 1991 *Proc. R. Soc.* A **434** 263–78
[23] Manganaro N and Parker D F 1993 *J. Phys. A: Math. Gen.* **26** 4093–106
[24] Tajiri M 1983 *J. Phys. Soc. Japan* **52** 1908–17
[25] Baumann G and Nonnenmacher T F 1987 *J. Math. Phys.* **28** 1250–60
[26] Chazy J 1911 *Acta Math.* **34** 317–85
[27] Bureau F J 1972 *Ann. Math.* **91** 163–281
[28] Boiti M and Pempinelli F 1980 *Nuovo Cimento* B **59** 40–58
[29] Gagnon L, Grammaticos B, Ramani A and Winternitz P 1989 *J. Phys. A: Math. Gen.* **22** 499–509
[30] Jimbo M and Miwa T 1981 *Physica* D **2** 407–48